

Pair production and solutions of the wave equation for particles with arbitrary spin^{*}

S.I. Kruglov[†]

*International Education Centre, 2727 Steeles Ave. W, # 202,
Toronto, ON M3J 3G9, Canada*

On leave from

*National Scientific Center of Particle and High Energy Physics,
M. Bogdanovich St. 153, Minsk 220010, Belarus*

February 7, 2008

Abstract

We investigate the theory of particles with arbitrary spin and anomalous magnetic moment in the Lorentz representation $(0, s) \oplus (s, 0)$, in an external constant and uniform electromagnetic field. We obtain the density matrix of free particles in pure spin states. The differential probability of pair producing particles with arbitrary spin by an external constant uniform electromagnetic field has been found using the exact solutions. We have calculated the imaginary and real parts of the Lagrangian in an electromagnetic field that takes into account vacuum polarization.

1 Introduction

The interest to the theory of relativistic particles with arbitrary spins is growing now. One of the reason is that SUSY models require superpartners, i.e. additional fields of

^{*}Invited talk at the international workshop "Lorentz Group, CPT, and Neutrinos", Zacatecas, Mexico, June 23-26, 1999.

[†]E-mail: skruglov23@hotmail.com

particles with higher spins. In particular, it is important to take into account particles with spin $3/2$ - gravitons for the theory of inflation of the universe [1]. The second aspect is that some of string models have the similar features as models of relativistic spinning particles [2]. It is also interesting to have particles with arbitrary fractional spins [3] (see also [4, 5]). Such spinning particles in $2 + 1$ dimensions, called anyons, were discovered and they have anomalous statistics.

There are many different relativistic wave equations which describe particles with arbitrary spin [6-11]. The fields of free particles of a mass m and spin s in these formalisms realize definite representations of the Poincaré group. Some of these schemes for particles are equivalent each other. If the interaction with external electromagnetic fields is introduced, approaches based on different representations of the Lorentz group become not equivalent. Most of theories of particles in the external electromagnetic fields have difficulties such as non-causally propagation [12], indefinite metrics in the second-quantized theory [13-15] and others. We proceed here from the second order of the relativistic wave equation for particles with arbitrary spin s and magnetic moment μ on the base of the Lorentz representation $(0, s) \oplus (s, 0)$, where \oplus means the direct sum. Such an approach avoids some difficulties as compared with others schemes and the corresponding wave function has the minimal number of components. This is a generalization of the Feynman-Gell-Mann equation [16] for particles with spin $1/2$ in the case of higher spin particles which possess arbitrary magnetic moment. In particular case of spin $1/2$ we arrive to the Dirac theory. If the normal magnetic moment is considered, it leads to the approach [11]. Particles in this scheme propagate causally in external electromagnetic fields and this is a parity-symmetric theory with a Lagrangian formulation. The authors of the works [17] emphasized the importance of approaches based on the $(0, s) \oplus (s, 0)$ representation of the Lorentz group in the light of new experimental data. In [11] $(6s + 1)$ -component first-order matrix formulation of the equation for particles with arbitrary spin was considered. Then the author received the second order equation for particles with "normal" magnetic moment. Starting with the second order equation for particles which possess arbitrary magnetic moment we transit to the first order wave equation with another representation of the Lorentz group. But algebraic properties of the matrices are the same as in the approach [11]. We found solutions of equations for free particles in the form of density-matrices (projective matrices-dyads) for pure spin states which are used for different electromagnetic calculations of the Feynman diagrams. Such projective matrices-dyads allow to make covariant calculations without using matrices of the wave equation in the definite representation.

The main purpose of this paper is to investigate solutions of wave equations, pair production of arbitrary spin particles by constant uniform electromagnetic fields and vacuum

polarization of higher spin particles. Considering one-particle theory and obtaining the differential probability for pair production of particles with arbitrary spins we avoid the Klein paradox [18, 19]. As particular case of spin 1/2 and gyromagnetic ratio 2 particles we arrive at the well-known result found by Schwinger [22] who predicted e^+e^- pair production in the strong external electromagnetic field. It is actually now in the light of the development of power laser techniques. It should be noted that the pair production of particles by a gravitational field also is important for understanding the evolution of the universe near singularity [20].

The probability of pair production of particles in external electromagnetic fields can be found on the base of exact solutions of the wave equations [21] or the imaginary part of the Lagrangian [22]. We consider here both approaches. Nonlinear corrections to the Maxwell Lagrangian of the constant uniform electromagnetic fields are determined from the polarization of the vacuum of arbitrary spin particles. The problem of pair production of particles with higher spins using the quasiclassical scheme (method of "imaginary time") was considered in [23] which is agreed with our approach of the relativistic wave equations. It should be noted that the quasiclassical approximation has a restriction for the fields considered $E, H \ll m^2/e$ when the process is exponentially suppressed. It means that the approach [23] is valid when electromagnetic fields are not too strong and less than the critical value m^2/e . But it is known that pair of particles are created rapidly at the critical value of the fields. In our consideration there are no such restrictions. The problem of the pair production of vector particles with gyromagnetic ratio 2 was investigated in [24]. In [25, 26] the imaginary part of the effective Lagrangian which defines the probability of e^+e^- -production was found with taking into account anomalous magnetic moment.

We use system of units $\hbar = c = 1$, $\alpha = e^2/4\pi = 1/137$, $e > 0$. In section 2 proceeding from the second order equation for arbitrary spin particles with anomalous magnetic moment we transfer to the first order formulation of the theory. All independent solutions of the equation for free particles are found in the form of matrices-dyads (density matrix). Section 3 contains investigating of exact solutions of arbitrary spin particle equations in the constant uniform electromagnetic fields. We find on the base of exact solutions the differential probability for pair production of particles with the arbitrary spin and anomalous magnetic moment. The imaginary part of the effective Lagrangian for the electromagnetic fields is calculated. In section 4 using the Schwinger method we find the nonlinear corrections to the Lagrangian of a constant uniform electromagnetic fields, caused by the vacuum polarization of particles with the arbitrary spin and magnetic moment. Section 5 contains the discussion of results.

2 Wave Equation and Density Matrix

We proceed here from the Bargmann-Wightman-Wigner-type (BWW-type) quantum field theory [27, 17] based on the $(s, 0) \oplus (0, s)$ Lorentz representation for massive particles. In the BWW-type theories, a boson and antiboson have opposite intrinsic parities [28] and well-defined C and T characteristics [17]. The wave function of the $(s, 0) \oplus (0, s)$ representation has $2(2s + 1)$ component. For the spin 1/2 we arrive at well-known Dirac bispinors. But for spin 1 there is doubling of the component as compared with the Proca theory [29] because the vector particles have 3 spin states with the projections $s_z = \pm 1, 0$. As pointed in [17] the $(s, 0) \oplus (0, s)$ representation for massive particles opens new experimentally observable possibilities.

We postulate the next two (for $\varepsilon = \pm 1$) wave equations for arbitrary spin particles in external electromagnetic fields:

$$(D_\mu^2 - m^2 - \frac{eq}{2s} F_{\mu\nu} \Sigma_{\mu\nu}^{(\varepsilon)}) \Psi_\varepsilon(x) = 0, \quad (1)$$

where s is the spin of particles, $D_\mu = \partial_\mu - ieA_\mu$; $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the strength tensor, $\varepsilon = \pm 1$; and $\Sigma_{\mu\nu}^{(-)}$, $\Sigma_{\mu\nu}^{(+)}$ are the generators of the Lorentz group which correspond to the $(s, 0)$ and $(0, s)$ representations. Two equations (1) (for $\varepsilon = \pm 1$) describe particles which possess the magnetic moment $\mu = eq/(2m)$ and gyromagnetic ratio $g = q/s$. At $q = 1$ we have "normal" magnetic moment $\mu = e/(2m)$ and $g = 1/s$. Generators $\Sigma_{\mu\nu}^{(\varepsilon)}$ are connected with the spin matrices S_k by the relationships $\Sigma_{ij}^{(\varepsilon)} = \varepsilon_{ijk} S_k$, $\Sigma_{4k}^{(\varepsilon)} = -i\varepsilon S_k$, where the parameter $\varepsilon = \pm 1$ corresponds to the Lorentz group representations $(s, 0)$ and $(0, s)$. As usual, relations

$$[S_i, S_j] = i\varepsilon_{ijk} S_k, \quad (S_1)^2 + (S_2)^2 + (S_3)^2 = s(s + 1) \quad (2)$$

are valid, where $i, j, k = 1, 2, 3$; ε_{ijk} is the Levi-Civita symbol.

At $q = 1$ equations (1) where considered in [11]. The theory of arbitrary spin particles based on equations (1) is causal in the presence of external electromagnetic fields. It is seen from the method [12] that equations remain hyperbolic and the characteristic surfaces are lightlike. Equations (1) are invariant at the parity operation. Indeed, at the parity inversion $\varepsilon \rightarrow -\varepsilon$ and the representation $(s, 0)$ transforms into $(0, s)$.

Now we consider the problem to formulate a first-order relativistic wave equation from the second order equation (1). This is convenient for some quantum electrodynamics calculations with polarized particles of arbitrary spins.

Let us introduce the matrix $\varepsilon^{A,B}$ which has the dimension $n \times n$, its elements consists of zeroes and only one element is unit on the crossing of row A and column B . So the

matrix element and multiplication of this matrices are

$$\left(\varepsilon^{A,B}\right)_{CD} = \delta_{AC}\delta_{BD}, \quad \varepsilon^{A,B}\varepsilon^{C,D} = \delta_{BC}\varepsilon^{A,D}, \quad (3)$$

where indices $A, B, C, D = 1, 2 \dots n$.

The six generators $\Sigma_{\mu\nu}^{(+)}$ (or $\Sigma_{\mu\nu}^{(-)}$) entering equation (1) have the dimension $2s+1$. Therefore the wave function $\Psi_+(x)$ (or $\Psi_-(x)$) of equation (1) possesses $2s+1$ components. Now we can introduce the $5(2s+1)$ -component wave function

$$\Psi_1(x) = \begin{pmatrix} \Psi_+(x) \\ -\frac{1}{m}D_\mu\Psi_+(x) \end{pmatrix} \quad (4)$$

so that $\Psi_1(x) = \{\Psi_A(x)\}$, $A = 0, \mu$; $\Psi_0 = \Psi_+(x)$, $\Psi_\mu = -\frac{1}{m}D_\mu\Psi_+(x)$ and $\Psi_+(x)$ realizes the Lorentz representation $(s, 0)$. It is not difficult to check that equation (1) for $\varepsilon = +1$ can be represented as the first-order equation

$$\left(\beta_\mu^{(+)}D_\mu + m\right)\Psi_1(x) = 0, \quad (5)$$

where $5(2s+1) \times 5(2s+1)$ - matrices

$$\beta_\mu^{(+)} = \left(\varepsilon^{0,\mu} + \varepsilon^{\mu,0}\right) \otimes I_{2s+1} - i\frac{q}{s}\varepsilon^{0,\nu} \otimes \Sigma_{\mu\nu}^{(+)}, \quad (6)$$

are introduced and \otimes is the direct multiplication, I_{2s+1} is the unit matrix of the dimension $2s+1$ and in (6) we imply the summation on index ν . Using properties (3) it is easy to check that 5-dimensional matrices $\beta_\mu^{DK} = \varepsilon^{0,\mu} + \varepsilon^{\mu,0}$ obey the Duffin-Kemmer algebra [30, 31]

$$\beta_\mu^{DK}\beta_\nu^{DK}\beta_\alpha^{DK} + \beta_\alpha^{DK}\beta_\nu^{DK}\beta_\mu^{DK} = \delta_{\mu\nu}\beta_\alpha^{DK} + \delta_{\alpha\nu}\beta_\mu^{DK}. \quad (7)$$

The wave function $\Psi_1(x)$ transforms on the $[(0, 0) \oplus (1/2, 1/2)] \otimes (s, 0)$ representation of the Lorentz group [32,33]. For the case $\varepsilon = -1$ we have the analogous equation

$$\left(\beta_\mu^{(-)}D_\mu + m\right)\Psi_2(x) = 0, \quad (8)$$

where

$$\beta_\mu^{(-)} = \left(\varepsilon^{\bar{0},\bar{\mu}} + \varepsilon^{\bar{\mu},\bar{0}}\right) \otimes I_{2s+1} - i\frac{q}{s}\varepsilon^{\bar{0},\bar{\nu}} \otimes \Sigma_{\mu\nu}^{(-)},$$

$$\Psi_2(x) = \begin{pmatrix} \Psi_-(x) \\ -\frac{1}{m}D_\mu\Psi_-(x) \end{pmatrix} \quad (9)$$

and $\Psi_2(x) = \{\Psi_B(x)\}$, $B = \bar{0}, \bar{\mu}$; $\Psi_{\bar{0}}(x) = \Psi_-(x)$, $\Psi_{\bar{\mu}} = -\frac{1}{m}D_\mu\Psi_-(x)$ where $\Psi_-(x)$ transforms as $(0, s)$ representation of the Lorentz group and $\Psi_2(x)$ realizes the representation $[(0, 0) \oplus (1/2, 1/2)] \otimes (0, s)$. The two equations (5) and (7) can be combined into one first-order equation

$$(\beta_\mu D_\mu + m)\Psi(x) = 0 \quad (10)$$

with the matrices and wave function

$$\beta_\mu = \beta_\mu^{(+)} \oplus \beta_\mu^{(-)}, \quad \Psi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix}. \quad (11)$$

Using the properties of elements of the entire algebra (3), it is not difficult to get the relation equation for matrices β_μ (the same for $\beta_\mu^{(+)}$ and $\beta_\mu^{(-)}$):

$$\begin{aligned} & \beta_\mu\beta_\nu\beta_\sigma + \beta_\nu\beta_\sigma\beta_\mu + \beta_\sigma\beta_\mu\beta_\nu + \beta_\nu\beta_\mu\beta_\sigma + \beta_\mu\beta_\sigma\beta_\nu + \beta_\sigma\beta_\nu\beta_\mu = \\ & = 2(\delta_{\nu\sigma}\beta_\mu + \delta_{\mu\sigma}\beta_\nu + \delta_{\mu\nu}\beta_\sigma). \end{aligned} \quad (12)$$

In works [11] the $(6s+1)$ -dimensional representation of $SL(2, C)$ group was considered for particles of arbitrary spins with algebra (12). It is seen from (12) that the algebra of β_μ matrices is more complicated than Duffin-Kemmer algebra (7). Different representations of this algebra were studied in [34, 35].

Let us consider the problem to find the solutions to equation (10) for the definite momentum and spin projections. It is convenient to find these solutions in the form of projective matrix-dyads (density matrix). All electrodynamics calculations of Feynman diagrams with arbitrary spin particles can be done using these matrices. As particles in initial and final states are free particles, we can put parameter $q = 0$ in (1), (10). It corresponds to the case when external electromagnetic fields are absent. Then matrices β_μ transforms to β_μ^0 :

$$\beta_\mu^0 = [(\varepsilon^{0,\mu} + \varepsilon^{\mu,0}) \otimes I_{2s+1}] \oplus [(\varepsilon^{\bar{0},\bar{\mu}} + \varepsilon^{\bar{\mu},\bar{0}}) \otimes I_{2s+1}], \quad (13)$$

which obey the Duffin-Kemmer algebra (7). The projective operators extracting states with definite 4-momentum p_μ for particle and antiparticle are given by

$$\Lambda_\pm = \frac{i\hat{p}(i\hat{p} \pm m)}{2m}, \quad (14)$$

where $\hat{p} = p_\mu\beta_\mu^0$ (we use the metric such that $p^2 = \mathbf{p}^2 + p_4^2 = \mathbf{p}^2 - p_0^2 = -m^2$). Signs $+$ and $-$ in (14) correspond to the particle and antiparticle, respectively. Matrices Λ_\pm have the usual projective operator properties [36]

$$\Lambda_\pm^2 = \Lambda_\pm. \quad (15)$$

Equation (15) is checked by the relation $\hat{p}^3 = p^2 \hat{p}$ which follows from the Duffin-Kemmer algebra (7). To find the spin projective operators we need the generators of the Lorentz group in the representation of the wave function $\Psi(x)$ which enters equation (10). From the structure of the functions $\Psi_1(x)$, $\Psi_2(x)$ (4), (9) we define the generators of the Lorentz group in our $10(2s+1)$ -dimension representation

$$\begin{aligned} J_{\mu\nu} &= J_{\mu\nu}^{(+)} \oplus J_{\mu\nu}^{(-)}, \\ J_{\mu\nu}^{(+)} &= (\varepsilon^{\mu,\nu} - \varepsilon^{\nu,\mu}) \otimes I_{2s+1} + iI_5 \otimes \Sigma_{\mu\nu}^{(+)}, \\ J_{\mu\nu}^{(-)} &= (\varepsilon^{\bar{\mu},\bar{\nu}} - \varepsilon^{\bar{\nu},\bar{\mu}}) \otimes I_{2s+1} + iI_5 \otimes \Sigma_{\mu\nu}^{(-)}, \end{aligned} \quad (16)$$

where I_5 is the 5-dimensional unit matrix. Using properties (3), we get the commutation relations

$$[J_{\mu\nu}, J_{\alpha\beta}] = \delta_{\nu\alpha} J_{\mu\beta} + \delta_{\mu\beta} J_{\nu\alpha} - \delta_{\mu\alpha} J_{\nu\beta} - \delta_{\nu\beta} J_{\mu\alpha}, \quad (17)$$

$$[\beta_\mu, J_{\alpha\beta}] = \delta_{\mu\alpha} \beta_\beta - \delta_{\mu\beta} \beta_\alpha. \quad (18)$$

The relationship (17) is a well-known commutation relation for generators of the Lorentz group [32, 33]. Equation (10) is a form invariant under the Lorentz transformations because relation (18) is valid. To guarantee the existence of a relativistically invariant bilinear form

$$\bar{\Psi}\Psi = \Psi^+ \eta \Psi, \quad (19)$$

where Ψ^+ is the Hermite conjugated wave function, we should construct a Hermitianizing matrix η with the properties [34, 36]:

$$\eta\beta_i = -\beta_i\eta, \quad \eta\beta_4 = \beta_4\eta \quad (i = 1, 2, 3). \quad (20)$$

Such matrix exists and is given by

$$\eta = (\varepsilon^{a,\bar{a}} + \varepsilon^{\bar{a},a} - \varepsilon^{4,\bar{4}} - \varepsilon^{\bar{4},4} - \varepsilon^{0,\bar{0}} - \varepsilon^{\bar{0},0}) \otimes I_{2s+1}, \quad (21)$$

where the summation on the index $a = 1, 2, 3$ is implied. Now we introduce the operator of the spin projection on the direction of the momentum \mathbf{p} :

$$\hat{S}_{\mathbf{p}} = -\frac{i}{2|\mathbf{p}|} \varepsilon_{abc} p_a J_{bc} = (\kappa_p + \sigma_p) \oplus (\bar{\kappa}_p + \bar{\sigma}_p), \quad (22)$$

where

$$\begin{aligned}\kappa_p &= -\frac{i}{|\mathbf{p}|} \varepsilon_{abc} p_a \varepsilon^{b,c} \otimes I_{2s+1}, & \bar{\kappa}_p &= -\frac{i}{|\mathbf{p}|} \varepsilon_{abc} p_a \varepsilon^{\bar{b},\bar{c}} \otimes I_{2s+1}, \\ \sigma_p &= \bar{\sigma}_p = I_5 \otimes \frac{\mathbf{p}\mathbf{S}}{|\mathbf{p}|},\end{aligned}\tag{23}$$

and $|\mathbf{p}| = \sqrt{p_1^2 + p_2^2 + p_3^2}$. It is easy to check that the required relation holds

$$[\hat{S}_{\mathbf{p}}, \hat{p}] = 0.$$

The matrices $\kappa_p, \bar{\kappa}_p$ obey the simple equations

$$\kappa_p^3 = \kappa_p, \quad \bar{\kappa}_p^3 = \bar{\kappa}_p.\tag{24}$$

Taking into account equations (2) we write out relations for the matrices σ_p (23):

$$\begin{aligned}\left(\sigma_p^2 - \frac{1}{4}\right) \cdots \left(\sigma_p^2 - s^2\right) &= 0 & \text{for odd spins,} \\ \sigma_p \left(\sigma_p^2 - 1\right) \cdots \left(\sigma_p^2 - s^2\right) &= 0 & \text{for even spins.}\end{aligned}\tag{25}$$

Relations (25) allow us to construct projective operators which extract the pure spin states. Using the relationship

$$\hat{S}_{\mathbf{p}} \hat{p} = (\sigma_p \oplus \bar{\sigma}_p) \hat{p}$$

we can consider projective matrices on the base of equations (25). The common technique of the construction of such operators is described in [36]. Let us consider the equation for auxiliary spin operators $\sigma_p, \bar{\sigma}_p$ for spin projection s_k :

$$\sigma_p \Psi_k = s_k \Psi_k.\tag{26}$$

The solution to equation (26) can be found using relationships (25) which can be rewritten as

$$(\sigma_p - s_k) P_k(s) = 0,\tag{27}$$

where polynomials $P_k(s)$ are given by

$$P_k(s) = \left(\sigma_p^2 - \frac{1}{4}\right) \cdots (\sigma_p + s_k) \cdots (\sigma_p^2 - s^2) \quad \text{for odd spins,}$$

$$P_k(s) = \sigma_p (\sigma_p^2 - 1) \cdots (\sigma_p + s_k) \cdots (\sigma_p^2 - s^2) \quad \text{for even spins.} \quad (28)$$

Every columns of the polynomial $P_k(s)$ can be considered as eigenvector Ψ_k of equation (26) with the eigenvalue s_k . As s_k is one multiple root of equations (25), all columns of the matrix $P_k(s)$ are linear independent solutions of equation (26) [36]. Using definitions (28) it can be verified that matrix

$$Q_k = \frac{P_k(s)}{P_k(s_k)} \quad (29)$$

is the projective operator with the relation

$$Q_k^2 = Q_k. \quad (30)$$

Equation (30) tells that the matrix Q_k can be transformed into diagonal form, where on the diagonal there are only units and zeroes. So the Q_k acting on the wave function Ψ will remain components which correspond to the spin projection s_k .

We have mentioned that this theory of arbitrary spin particles has doubling of spin states of particles because there are two representations $(s, 0)$ and $(0, s)$ of the Lorentz group. To differ this representations (for $s > 1/2$) which are connected by the parity transformations we use the parity operator

$$K = (\varepsilon^{\mu, \bar{\mu}} + \varepsilon^{\bar{\mu}, \mu} + \varepsilon^{0, \bar{0}} + \varepsilon^{\bar{0}, 0}) \otimes I_{2s+1} \quad (31)$$

with the summation on indices $\mu = 1, 2, 3, 4$. The $10(2s+1)$ matrix K has the property $K^2 = I_{10(2s+1)}$. The projective operator extracting states with the definite parity is given by

$$M_\varepsilon = \frac{1}{2} (K + \varepsilon), \quad (32)$$

where $\varepsilon = \pm 1$. This matrix possesses the required relationship

$$M_\varepsilon^2 = M_\varepsilon. \quad (33)$$

It should be noted that the matrix K (31) plays the role analogous to the γ_5 -matrix in the Dirac theory of particles with the spin $1/2$. It is checked that operators \hat{p} , $\hat{S}_{\mathbf{p}}$, K commute each other and as a consequence, they have the common eigenvector. The projective operator extracting pure state with the definite 4-momentum projections, spins and parity is given by

$$\Pi_{\pm m, k, \varepsilon} = \Lambda_{\pm} M_{\varepsilon} (Q_k \oplus Q_k) \quad (34)$$

with matrices (14), (29) and (32). The $\Pi_{\pm m, k, \varepsilon}$ is the density matrix for pure states. It is easy to consider not pure states by summing (34) over definite quantum numbers s_k, ε . Projective operator for pure states can be represented as matrix-dyad [36]:

$$\Pi_{\pm m, k, \varepsilon} = \Psi_{\pm m, k}^{\varepsilon} \cdot \bar{\Psi}_{\pm m, k}^{\varepsilon}, \quad (35)$$

where $\bar{\Psi}_{\pm m, k}^{\varepsilon} = (\Psi_{\pm m, k}^{\varepsilon})^+ \eta$ and the $\Psi_{\pm m, k}^{\varepsilon}$ is the solution to equations

$$(i\beta_{\mu} p_{\mu} \pm m) \Psi_{\pm m, k}^{\varepsilon} = 0, \quad \hat{S}_{\mathbf{p}} \Psi_{\pm m, k}^{\varepsilon} = s_k \Psi_{\pm m, k}^{\varepsilon},$$

$$K \Psi_{\pm m, k}^{\varepsilon} = \varepsilon \Psi_{\pm m, k}^{\varepsilon}. \quad (36)$$

Expression (35) is convenient for calculations of different quantum electrodynamics processes with polarized particles of arbitrary spins.

3 Pair Production by External Electromagnetic Fields

To calculate the probability of pair production of arbitrary spin particles, we follow the Nikishov method [21]. So exact solutions to equation (1) should be found for external constant uniform electromagnetic fields. Using the properties of generators $\Sigma_{\mu\nu}^{(\varepsilon)}$ we find the relationships

$$\frac{1}{2} \Sigma_{\mu\nu}^{(+)} F_{\mu\nu} = S_i X_i, \quad \frac{1}{2} \Sigma_{\mu\nu}^{(-)} F_{\mu\nu} = S_i X_i^*, \quad (37)$$

where $X_i = H_i + iE_i$, $X_i^* = H_i - iE_i$; E_i, H_i are the electric and magnetic fields, respectively, and the spin matrices S_i obey equations (2). In the diagonal representation, the equations for eigenvalues are given by

$$S_i X_i \Psi_+^{(\sigma)}(x) = \sigma X \Psi_+^{(\sigma)}(x), \quad S_i X_i^* \Psi_-^{(\sigma)}(x) = \sigma X^* \Psi_-^{(\sigma)}(x), \quad (38)$$

where $X = \sqrt{\mathbf{X}^2}$, $\mathbf{X} = \mathbf{H} + i\mathbf{E}$, and the spin projection σ is

$$\sigma = \begin{array}{l} \pm s, \pm(s-1), \dots, 0 \\ \pm s, \pm(s-1), \dots, \pm \frac{1}{2} \end{array} \quad \begin{array}{l} \text{for even spins,} \\ \text{for odd spins.} \end{array} \quad (39)$$

Taking into account (37), (38), equations (1) (for $\varepsilon = \pm 1$) are rewritten as

$$(D_\mu^2 - m^2 - a\sigma X)\Psi_+^{(\sigma)}(x) = 0, \quad (D_\mu^2 - m^2 - a\sigma X^*)\Psi_-^{(\sigma)}(x) = 0, \quad (40)$$

where $a = eq/s$. These equations are like the Klein-Gordon equation for scalar particles but with complex "effective" masses: $m_{eff}^2 = m^2 + a\sigma X$, $(m_{eff}^2)^* = m^2 + a\sigma X^*$. It is sufficient to consider only one of equations (40). Let us consider the solution of the equation

$$(D_\mu^2 - m_{eff}^2)\Psi^{(\sigma)}(x) = 0, \quad (\Psi^{(\sigma)}(x) \equiv \Psi_+^{(\sigma)}(x)) \quad (41)$$

in the presence of the external constant uniform electromagnetic fields. The general case is when two Lorentz invariants of the electromagnetic fields $\mathcal{F} = \frac{1}{4}F_{\mu\nu}^2 \neq 0$, $\mathcal{G} = \frac{1}{4}F_{\mu\nu}\tilde{F}_{\mu\nu} \neq 0$ ($\tilde{F}_{\mu\nu} = \frac{i}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$, $\epsilon_{\mu\nu\alpha\beta}$ is the antisymmetric Levi-Civita tensor). Then there is a coordinate system where the electric \mathbf{E} and magnetic \mathbf{H} fields are parallel, i.e. $\mathbf{E} \parallel \mathbf{H}$. In this case the 4-vector potential is given by

$$A_\mu = (0, x_1 H, -x_0 E, 0) \quad (42)$$

so that 3-vectors $\mathbf{E} = \mathbf{n}E$, $\mathbf{H} = \mathbf{n}H$ directed along the axes 3, where $\mathbf{n} = (0, 0, 1)$ is a unit vector. The four solutions of equation (41) for the potential (42) with different asymptotic are given by [21, 37] (see also [38])

$$\pm \Psi_{p,n}^{(\sigma)}(x) = N \exp \left\{ i(p_2 x_2 + p_3 x_3) - \frac{\eta^2}{2} \right\} H_n(\eta) \pm \psi^{(\sigma)}(\tau) \quad (43)$$

where N is the normalization constant, $H_n(\eta)$ is the Hermite polynomial,

$$\eta = \frac{eHx_1 + p_2}{\sqrt{eH}}, \quad \nu = \frac{ik^2}{2eE} - \frac{1}{2}, \quad \tau = \sqrt{eE} \left(x_0 + \frac{p_0}{eE} \right)$$

and

$$\begin{aligned} {}_+\psi^{(\sigma)}(\tau) &= D_\nu[-(1-i)\tau] & {}_-\psi^{(\sigma)}(\tau) &= D_\nu[(1-i)\tau] \\ {}_+\psi^{(\sigma)}(\tau) &= D_{\nu^*}[(1+i)\tau] & {}_-\psi^{(\sigma)}(\tau) &= D_{\nu^*}[-(1+i)\tau] \end{aligned} \quad (44)$$

Here $D_\nu(x)$ is the Weber-Hermite function (the parabolic-cylinder function). The probability for pair production of particles with arbitrary spins by the constant electromagnetic fields can be obtained through the asymptotic of solutions (44) when the time $x_0 \rightarrow \pm\infty$. The functions ${}_+\psi^{(\sigma)}(\tau)$ have positive frequency at $x_0 \rightarrow \pm\infty$ and ${}_-\psi^{(\sigma)}(\tau)$ -

negative frequency. The constant k^2 which enters the index ν of the parabolic-cylinder functions (44) is given by [38]

$$k^2 = m_{eff}^2 + eH(2n + 1), \quad (45)$$

where $n = l + r$, l is the orbital quantum number, r is the radial quantum number and $n = 0, 1, 2, \dots$ is the principal quantum number. It should be noted that for scalar particles we have the equation $k^2 = p_0^2 - p_3^2$, where p_0 is the energy and p_3 is the third projection of the momentum of a scalar particle. In our case of arbitrary spin particles, the parameter m_{eff}^2 is the complex value. Nevertheless all physical quantities in this case are the real values. Solutions (43), (44) are characterized by three conserved numbers: k^2 and the momentum projections p_2 , p_3 . As shown in [21] functions (43) are connected by the relations

$$\begin{aligned} {}^+\Psi_{p,n}^{(\sigma)}(x) &= c_{1n\sigma} {}^+\Psi_{p,n}^{(\sigma)}(x) + c_{2n\sigma} {}^-\Psi_{p,n}^{(\sigma)}(x), \\ {}^+\Psi_{p,n}^{(\sigma)}(x) &= c_{1n\sigma}^* {}^+\Psi_{p,n}^{(\sigma)}(x) - c_{2n\sigma} {}^-\Psi_{p,n}^{(\sigma)}(x), \\ {}^-\Psi_{p,n}^{(\sigma)}(x) &= -c_{2n\sigma}^* {}^+\Psi_{p,n}^{(\sigma)}(x) + c_{1n\sigma} {}^-\Psi_{p,n}^{(\sigma)}(x), \\ {}^-\Psi_{p,n}^{(\sigma)}(x) &= c_{2n\sigma}^* {}^+\Psi_{p,n}^{(\sigma)}(x) + c_{1n\sigma}^* {}^-\Psi_{p,n}^{(\sigma)}(x), \end{aligned} \quad (46)$$

where coefficients $c_{1n\sigma}$, $c_{2n\sigma}$ are given by

$$\begin{aligned} c_{2n\sigma} &= \exp \left[-\frac{\pi}{2}(\lambda + i) \right] & \lambda &= \frac{m_{eff}^2 + eH(2n + 1)}{eE}, \\ |c_{1n\sigma}|^2 - |c_{2n\sigma}|^2 &= 1 & \text{for even spins,} \\ |c_{1n\sigma}|^2 + |c_{2n\sigma}|^2 &= 1 & \text{for odd spins.} \end{aligned} \quad (47)$$

The values $c_{1n\sigma}$, $c_{2n\sigma}$ are connected with the probability of the pair producing of arbitrary spin particles in the state with the quantum number n and the spin projection σ . The absolute probability for a production of a pair in the state with quantum number n , momentum p and the spin projection σ in all space and during all time is

$$|c_{2n\sigma}|^2 = \exp \left\{ -\pi \left[\frac{m^2}{eE} + \frac{q\sigma H}{sE} + \frac{H}{E}(2n + 1) \right] \right\}. \quad (48)$$

The value (48) is also the probability of the annihilation of a pair with quantum numbers n , p , σ with the energy transfer to the external electromagnetic fields. It is seen from (48) that for $H \gg E$ the pair of particles are mainly created by the external

fields in the state with $n = 0$, $\sigma = -s$. This is the state with the smallest energy. So at $H \gg E$ there is a production of polarized beams of particles and antiparticles with the spin projection $\sigma = -s$. (the s is the spin of particles). The average number of produced pairs of particles from a vacuum is

$$\overline{N} = \int \sum_{n,\sigma} |c_{2n\sigma}|^2 dp_2 dp_3 \frac{L^2}{(2\pi)^2} \quad (49)$$

because $(2\pi)^{-2} dp_2 dp_3 L^2$ is the density of final states, where the L is the cut-off along the coordinates, so the L^3 is the normalization volume. The variables η, τ define the region of forming the process which is described by solutions (43) with the coordinates of the centre of this region $x_0 = -p_3/eE$, $x_1 = -p_2/eH$. Therefore instead of the integration over p_2 and p_3 in (49) it is possible to use the substitution [21]

$$\int dp_2 \rightarrow eHL, \quad \int dp_3 \rightarrow eET \quad (50)$$

with the time of observation T .

Calculating the sum in (49) over the principal quantum number n , with the help of (48), (50) we obtain the probability of the pair production per unit volume and per unit time

$$I(E, H) = \frac{\overline{N}}{VT} = \frac{e^2 EH}{8\pi^2} \frac{\exp[-\pi m^2/(eE)]}{\sinh(\pi H/E)} \sum_{\sigma} \exp(-\sigma b), \quad (51)$$

where $b = \pi qH/(sE)$, $V = L^3$. Now we calculate the sum over the spin projection σ in (51) for odd and even spins:

1) even spins

$$\sum_{\sigma} \exp(-\sigma b) = S_1 + S_2, \quad S_1 = e^0 + e^{-b} + \dots + e^{-sb} = \frac{e^{-b(s+1)} - 1}{e^{-b} - 1},$$

$$S_2 = e^b + e^{2b} + \dots + e^{sb} = \frac{e^b(e^{bs} - 1)}{e^b - 1}, \quad S_1 + S_2 = \frac{\cosh(sb) - \cosh[(s+1)b]}{1 - \cosh b} \quad (52)$$

2) odd spins

$$\sum_{\sigma} \exp(-\sigma b) = S'_1 + S'_2, \quad S'_1 = e^{-b/2} + e^{-3b/2} + \dots + e^{-sb} = \frac{e^{-b(s+1)} - e^{-b/2}}{e^{-b} - 1},$$

$$S'_2 = e^{b/2} + e^{3b/2} + \dots + e^{sb} = \frac{e^{b(s+1)} - e^{b/2}}{e^b - 1}, S'_1 + S'_2 = \frac{\cosh(sb) - \cosh[(s+1)b]}{1 - \cosh b} \quad (53)$$

So we get the same final expressions for even and odd spins. Using the relationship

$$\frac{\cosh(sb) - \cosh[(s+1)b]}{1 - \cosh b} = \cosh(sb) + \sinh(sb) \coth \frac{b}{2} = \frac{\sinh[b(s+1/2)]}{\sinh(b/2)} \quad (54)$$

and equations (51), (52) we arrive at the pair-production probability

$$I(E, H) = \frac{\overline{N}}{VT} = \frac{e^2 EH \exp[-\pi m^2/(eE)] \sinh[(2s+1)q\pi H/(2sE)]}{8\pi^2 \sinh(\pi H/E) \sinh[q\pi H/(2sE)]}. \quad (55)$$

Expression (55) coincides with those derived by [23] using the quasiclassical approach. So the $I(E, H)$ is the intensity of the creation of pairs of arbitrary spin particles which possess the magnetic moment $\mu = eq/(2m)$ and gyromagnetic ratio $g = q/s$.

In [23] there is a discussion of physical consequences which follow from equation (55). In particular, there is a pair production in pure magnetic field if $q = gs > 1$ [23]. It is interesting that the exact formula derived here from quantum field theory which is valid for arbitrary fields E, H , coincides with the asymptotic expression obtained by [23] for $E, H \ll m^2/e$.

To get the imaginary part of the density of the Lagrangian we use the relationship [21]

$$VTIm\mathcal{L} = \frac{1}{2} \int \sum_{n,\sigma} \ln |c_{1n\sigma}|^2 dp_2 dp_3 \frac{L^2}{(2\pi)^2}. \quad (56)$$

With the help of (47), (50) we arrive at (see also [23])

$$Im\mathcal{L} = \frac{e^2 EH}{16\pi^2} \sum_{n=1}^{\infty} \frac{\beta_n}{n} \exp\left(-\frac{\pi m^2 n}{eE}\right) \frac{\sinh[n(2s+1)q\pi H/(2sE)]}{\sinh(n\pi H/E) \sinh[nq\pi H/(2sE)]}, \quad (57)$$

where

$$\beta_n = \begin{cases} (-1)^{n-1} & \text{for bosons} \\ 1 & \text{for fermions.} \end{cases}$$

The different expressions for bosons and fermions occur due to different statistics and relations (47). The first term ($n = 1$) in (57) coincides with the probability of the pair production per unit volume per unit time divided by 2 [22] (see discussion in [23]).

4 Vacuum Polarization of Arbitrary Spin Particles

Now we calculate the nonlinear corrections to the Lagrangian of a constant uniform electromagnetic field interacting with the vacuum of arbitrary spin particles with the gyromagnetic ratio g . For the case of spins 0, 1/2 and 1 (for $g = 2$) such problem was solved by authors [39, 40, 22, 24]. The nonlinear corrections to Lagrangian of the electromagnetic field describe the effect of scattering of light by light. We consider one loop corrections corresponding to arbitrary spin particles to the Maxwell Lagrangian. For this purpose to take into account the vacuum polarization, it is convenient to explore the Schwinger method [22]. Applying this approach to the arbitrary spin particles described by equation (1) we arrive at the effective Lagrangian of constant uniform electromagnetic fields

$$\mathcal{L}_1 = -\frac{1}{32\pi^2} \int_0^\infty d\tau \tau^{-3} \exp(-m^2\tau - l(\tau)) \operatorname{tr} \exp\left(\frac{eq}{2s} \Sigma_{\mu\nu} F_{\mu\nu} \tau\right), \quad (58)$$

where

$$\Sigma_{\mu\nu} = \Sigma_{\mu\nu}^{(+)} \oplus \Sigma_{\mu\nu}^{(-)}, \quad l(\tau) = \frac{1}{2} \operatorname{tr} \ln \left[(eF\tau)^{-1} \sin(eF\tau) \right] \quad (59)$$

and $F_{\mu\nu}$ is a constant tensor of electromagnetic fields. The formal different of (60) from the case of spin 1/2 particles is in the substitution $\sigma_{\mu\nu} \rightarrow (q/s)\Sigma_{\mu\nu}$, where $\sigma_{\mu\nu} = (i/2) [\gamma_\mu, \gamma_\nu]$, γ_μ are the Dirac matrices. The problem is to calculate the trace of matrices entering the exponential factor in (58). Using relations (37)-(39), (52)-(54) we find

$$\operatorname{tr} \exp\left(\frac{eq}{2s} \Sigma_{\mu\nu} F_{\mu\nu} \tau\right) = 2 \operatorname{Re} \left[\cosh(eqX\tau) + \sinh(eqX\tau) \coth\left(\frac{eqX\tau}{2s}\right) \right]. \quad (60)$$

Inserting (61) into (60) and adding the constant which is necessary to cancel \mathcal{L}_1 when electromagnetic fields are turned off (see [22]) we arrive at

$$\begin{aligned} \mathcal{L}_1 = & -\frac{1}{8\pi^2} \int_0^\infty d\tau \tau^{-3} \exp(-m^2\tau) \times \\ & \times \left[(e\tau)^2 \mathcal{G} \frac{\operatorname{Re} [\cosh(eqX\tau) + \sinh(eqX\tau) \coth(eqX\tau/(2s))]}{2Im \cosh(eX\tau)} - \frac{2s+1}{2} \right], \end{aligned} \quad (61)$$

where $\mathcal{G} = \mathbf{E}\mathbf{H}$. At $q = 1$ and $s = 1/2$ Lagrangian (61) coincides with the Schwinger one [22]. Expression (61) is the correction to the Maxwell Lagrangian with taking into account the vacuum polarization of arbitrary spin particles which possesses the magnetic moment $\mu = eq/(2m)$ and gyromagnetic ratio $g = q/s$. Adding (61) to the Lagrangian of the free electromagnetic fields

$$\mathcal{L}_0 = -\mathcal{F} = \frac{1}{2} (\mathbf{E}^2 - \mathbf{H}^2)$$

and introducing the divergent constant for weak fields, we get the expression for the total Maxwell Lagrangian

$$\mathcal{L}_M = \mathcal{L}_0 + \mathcal{L}_1 = -Z\mathcal{F} - \frac{1}{8\pi^2} \int_0^\infty d\tau \tau^{-3} \exp(-m^2\tau) \times \\ \times \left[(e\tau)^2 \mathcal{G} \frac{\text{Re} [\cosh(eqX\tau) + \sinh(eqX\tau) \coth(eqX\tau/(2s))]}{2Im \cosh(eX\tau)} - \frac{2s+1}{2} - 4\beta(e\tau)^2 \mathcal{F} \right], \quad (62)$$

where

$$Z = 1 + \frac{e^2\beta}{2\pi^2} \int_0^\infty d\tau \tau^{-1} \exp(-m^2\tau), \quad \beta = \frac{q^2(2s^2 + 3s + 1) - s(2s + 1)}{24s}. \quad (63)$$

Following the Schwinger procedure, the renormalization of electromagnetic fields $\mathcal{F} \rightarrow Z\mathcal{F}$ and charge $e \rightarrow Z^{-1/2}e$ are used. After expanding (63) in the small electric E and magnetic H fields we arrive at the Lagrangian of constant uniform electromagnetic fields (in rational units)

$$\mathcal{L}_M = \frac{1}{2} (\mathbf{E}^2 - \mathbf{H}^2) + \frac{2\alpha^2}{45m^4} \left[(\mathbf{E}^2 - \mathbf{H}^2)^2 (15\beta - \gamma) + (\mathbf{EH})^2 \left(4\gamma + \frac{2s+1}{2} \right) \right] + \dots \quad (64)$$

where $\alpha = e^2/(4\pi)$ and

$$\gamma = \frac{[q^4(6s^4 + 15s^3 + 10s^2 - 1) - 3s^3(2s + 1)]}{16s^3}. \quad (65)$$

It is easy to check that as a particular case at $s = 1/2$, $q = 1$ (which corresponds to the Dirac theory), (65) coincides with the well-known Schwinger Lagrangian [22]. At $s = 1$ and $q = 1$ expression (64) is different from one received in [24]. It is because the considered theory of particles at $s = 1$ is not equivalent to the Proca theory. There is the doubling of spin states here. Effective Lagrangian (65) is the Heisenberg-Euler type which has been found for the case of polarization vacuum of particles with arbitrary spin and magnetic moment. Here we took into account virtual arbitrary spin particles but not virtual photons. It is because at small energies of the external fields the radiative corrections are small quantities. It is not difficult to find the asymptotic of (62) for over critical fields $eE/m^2 \rightarrow \infty$ and $eH/m^2 \rightarrow \infty$. It should be noted, however, that for strong electromagnetic fields anomalous magnetic moment of electrons depend on external fields [41, 42] and so do for arbitrary spin particles. Therefore to make the correct limit it is necessary to take into account this dependence [26].

It is possible to receive also the imaginary part of Lagrangian (57) from (62) using the residue theorem taking into account of poles of expression (62) and passing above them [22].

5 Discussion of the Results

The theory of particles with arbitrary spins and magnetic moment based on equation (1) and the corresponding Lagrangian allow to find density matrices (34), (35), pair-production probability (55) and effective Lagrangian for electromagnetic fields (65) also taking into account the polarization of vacuum. It is convenient to use matrices-diads (34), (35) for different electrodynamics calculations with the presence of particles with arbitrary spins. The exact formula for the intensity of pair production of arbitrary spin particles coincides with the expression obtained by [23] using the quasiclassical method of "imaginary time" which is valid only for $E, H \ll m^2/e$, i.e. for weak fields. From this follows that the analysis made in [23] is valid for arbitrary electromagnetic fields and is grounded by the relativistic quantum field theory. In particular, there is a pair production by a pure magnetic field $gs > 1$ [23] and in the presence of the magnetic field the probability decreases for scalar particles and increases for higher spin particles. As all divergences and the renormalizability are contained in $Re\mathcal{L}$ (62) but not in $Im\mathcal{L}$, the pair production probability does not depend on scheme of renormalizability. The cases of scalar and spinor (with $g = 2$) particles are reliable for calculations of vacuum polarization corrections due to their theories are renormalizable. The general formula (62) obtained here presents interest for the further development of the field theory particles with higher spins (see discussion in [23], [24]). Expression (62) is a reasonable result for arbitrary values of s and q because for particular case of scalar and spinor particles we arrive at known result. So we have here a reasonable and non-contradictory description of the nonlinear effects that arise in this interaction.

References

- [1] G. Felder, L. Kofman, A. Linde, hep-ph/9903350.
- [2] I. L. Buchbinder and SH. M. Shvartsman, Int. J. Mod. Phys. **A8**, 683 (1993).
- [3] F. Wilczek, Fractional Statistics and Anyon Superconductivity (World Scientific, 1990).
- [4] J. L. Cortes and M. S. Plyushchay, Int. J. Mod. Phys. **A11**, 3331 (1996).
- [5] S. I. Kruglov, Mod. Phys. Lett., **A12**, 1699 (1997); Europhys. Lett. **44**, 361 (1998);
S. I. Kruglov and M. N. Sergeenko, Mod. Phys. Lett. **A12**, 2475 (1997).

- [6] H. J. Bhabha, Rev. Mod. Phys. **17**, 200 (1945).
- [7] M. Fierz and W. Pauli, Proc. Roy. Soc. (London) **A173**, 211 (1939).
- [8] W. Rarita, J. Schwinger, Phys. Rev. **60**, 61 (1941).
- [9] V. Bargman, E. Wigner, Proc. Am. Acad. Sci. **34**, 211 (1948).
- [10] M. A. K. Khalil, Progr. Theor. Phys. **58**, 1538 (1977).
- [11] W. J. Hurley. Phys. Rev. **D4** 3605 (1971); **D10** 1185 (1974); Phys. Rev. Lett. **29**, 1475 (1972).
- [12] G. Velo, D. Zwanziger, Phys. Rev. **186**, 1337 (1969).
- [13] K. Johnson and E. C. G. Sudarshan, Ann. Phys. **13**, 126 (1961).
- [14] W. K. Tung, Phys. Rev. Lett. **16**, 763 (1966); Phys. Rev. **156**, 1385 (1967).
- [15] S. J. Chang, Phys. Rev. Lett. **17**, 1024 (1966).
- [16] R. P. Feynman, M. Gell-Mann, Phys. Rev. **109**, 193 (1958).
- [17] D. V. Ahluwalia, T. Goldman, M. B. Jonson, Acta Phys. Pol. **B25**, 1267 (1994); Mod. Phys. Lett. **A9**, 439 (1994); D. V. Ahluwalia, M. B. Jonson, T. Goldman, Phys. Lett. **B316**, 102 (1993); D. V. Ahluwalia, Int. J. Mod. Phys. **A11**, 1855 (1996); V. V. Dvoeglasov, Int. J. Theor. Phys. **37**, 1915 (1998).
- [18] F. Sauter, Z. Physik **69**, 742; **73**, 547 (1931).
- [19] J. M. Jauch and F. Rohrlich, The Theory of Photons and Electrons (Addison-Wesley Publishing Co., Cambridge, Mass., 1955).
- [20] Ya. B. Zel'dovich, L. P. Pitaevskii, V. S. Popov and A. A. Starobinskii, Sov. Phys.-Usp. **14**, 811 (1972) (Usp. Fiz. Nauk **105** , 781 (1971)).
- [21] A. I. Nikishov, Sov. Phys.-JETP **30**, 660 (1970) (Zh. Eksp. Teor. Fiz. **57**, 1210 (1969)); Nucl. Phys. **B21**, 346 (1970); Trudy FIAN **111**, 152 (1979).
- [22] J. Schwinger. Phys. Rev. **82** 664 (1951).

- [23] M. S. Marinov and V. S. Popov, Sov. J. Nucl. Phys. **15**, 702 (1972) (Yad. Fiz. **15**, 1271 (1972)).
- [24] V. S. Vanyashin and M. V. Terent'ev, Sov. Phys.-JETP **21**, 375 (1965) (Zh. Eksp. Teor. Fiz. **48**, 565 (1965)).
- [25] W. Dittrich, J. Phys. A: Math. Gen. **11**, 1191 (1978).
- [26] P. M. Lavrov, Izv. Vuz. Fiz. **3**, 37 (1987).
- [27] E. P. Wigner, in Group Theoretical Concepts and Methods in Elementary Particle Physics - Lectures of the Istanbul Summer School of Theoretical Physics, 1962, edited by F. Gursey.
- [28] L. H. Ryder, Quantum Field Theory (Cambridge Univ. Press, Cambridge, 1987).
- [29] A. Proca, Compt. Rend. **202**, 1420 (1936).
- [30] R. J. Duffin, Phys. Rev. **54**, 1114 (1938).
- [31] H. Kemmer, Proc. Roy. Soc. **173**, 91 (1939).
- [32] I. M. Gel'fand, R. A. Minlos and Z. Ya. Shapiro, Representations of the Rotation and Lorentz Groups and Their Applications (Pergamon, New York, 1963).
- [33] M. A. Naimark, Linear Representations of the Lorentz Group (Pergamon, New York, 1964).
- [34] Harish-Chandra, Phys. Rev. **71**, 793 (1947).
- [35] W. J. Hurley and E. C. G. Sudarshan, Ann. Phys. (N. Y.) **85**, 546 (1974).
- [36] F. I. Fedorov, Zh. Eksp. Teor. Fiz. **35**, 495 (1958); The Lorentz Group (Nauka, Moscow, 1979).
- [37] A. A. Grib, S. G. Mamayev and V. M. Mostepanenko, Quantum Effects in the Intensive External Fields (Moscow: Atomizdat).
- [38] S. I. Kruglov, J. Phys. G: Nucl. Phys. **21**, 1643 (1995); **22**, 461 (1996).
- [39] W. Heisenberg and H. Euler, Z. Physik **98**, 714 (1936).

- [40] V. Weisskopf, Kgl. Danske Videnskab, Selskab, Mat.-fys. Medd. **14**, No 6 (1936).
- [41] I. M. Ternov and V. G. Bagrov, Zh. Eksp. Teor. Fiz. **55**, 2273 (1968).
- [42] V. N. Bayer, B. M. Katkov and V. M. Strachovenko, Yad. Fiz. **24**, 379 (1976).